Numerical sequences and polynomials

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Abstract

In this paper, motivated by some problems of mathematical Olympiad caliber, we present links between numerical sequences and polynomials.

1 Basic Results

Hereafter, we present a basic general problem and solve it. Applying its solution, we will solve some other problems appeared elsewhere.

Problem 1 (Basic Problem). *For a given sequence $\{a_n\}_{n\geq 0}$ and for all $n \geq 0$, let $A_n(x)$ be the sequence of polynomials of degree at most n such that*

$$A_n(k) = a_k, \quad \textit{for } 0 \leq k \leq n.$$

Find the value of $A(n+1)$.

Solution. For $n = 0$ we have that $A_0(x) = \alpha_0$ is the constant polynomial and $A_0(1) = a_0$. So, $A_0(x) = a_0$ and $A_0(1) = a_0$.

For $n = 1$, let $A_1(x) = \alpha_0 + \alpha_1 x$. Since $A_1(0) = a_0$, then $A_1(x) = a_0 + \alpha_1 x$. On the other hand, $A_1(1) = a_1 \iff a_0 + \alpha_1 = a_1$ and $\alpha_1 = a_1 - a_0$. Thus, $A_1(x) = a_0 + (a_1 - a_0)x$, from which it follows that $A_1(2) = a_0 + 2(a_1 - a_0) = 2a_1 - a_0$.

For $n = 2$, let $A_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x(x-1)$. Since $A_2(x) = A_1(x)$ for $x = 0, 1$, then $A_1(x) = \alpha_0 + \alpha_1 x$ for all x and, therefore, $A_2(x) = A_1(x) + \alpha_2 x(x-1)$. Using that $A_2(2) = a_2$ and $A_1(2) = 2a_1 - a_0$ we obtain

$$2!\,\alpha_2 + 2a_1 - a_0 = a_2 \qquad \text{and} \qquad \alpha_2 = \frac{a_2 - 2a_1 + a_0}{2!}.$$

Thus,

$$\begin{aligned} A_2(x) &= A_1(x) + \frac{a_2 - 2a_1 + a_0}{2!}\,x(x-1) \\ &= a_0 + \frac{a_1 - a_0}{1!}x + \frac{a_2 - 2a_1 + a_0}{2!}\,x(x-1) \end{aligned}$$

and

$$A_2(3) = a_0 - 3a_1 + 3a_2.$$

To emphasize that $A_n(n+1)$ is not, generally speaking, a term of the sequence $\{a_n\}_{n\geq 0}$, we set $b_n = A_n(n+1)$ for all $n \geq 0$. Thus, our main goal is to express b_n through the terms of the sequence $\{a_n\}_{n\geq 0}$ and to find the polynomial $A_n(x)$. For example, we already have $b_0 = a_0$, $b_1 = 2a_1 - a_0$, $b_2 = a_0 - 3a_1 + 3a_2$ and, by the way, the polynomials $A_0(x)$, $A_1(x)$ and $A_2(x)$.

Now, assume that we already have the polynomials $A_0(x), A_1(x), \ldots, A_n(x)$ that satisfy the condition of the statement. We will find

$$A_{n+1}(x) = P(x) + \alpha_{n+1}\binom{x}{n+1},$$

where $\deg(P(x)) \leq n$ and

$$\binom{x}{n+1} = \frac{x(x-1)(x-2)\cdots(x-n)}{(n+1)!},$$

as is well-known.

Since $A_{n+1}(x) = A_n(x) = P(x)$ for $x = 0, 1, 2, \ldots, n$, then $P(x) = A_n(x)$ for all x and, therefore,

$$A_{n+1}(x) = A_n(x) + \alpha_{n+1}\binom{x}{n+1},$$

where coefficient α_{n+1} is determined by using $A_{n+1}(n+1) = \alpha_{n+1}$. We have

$$a_{n+1} = A_{n+1}(n+1) = A_n(n+1) + \alpha_{n+1}\binom{n+1}{n+1}$$

or

$$a_{n+1} = b_n + \alpha_{n+1} \Longleftrightarrow \alpha_{n+1} = a_{n+1} - b_n.$$

Thus,

$$A_{n+1}(x) = A_n(x) + (a_{n+1} - b_n)\binom{x}{n+1}.$$

Applying the $(n+1)$-times iterated difference operator Δ^{n+1} to the polynomial $A_{n+1}(x) = A_n(x) + \alpha_{n+1}\binom{x}{n+1}$, we obtain

$$\Delta^{n+1}(A_{n+1}(x)) = \Delta^{n+1}(A_n(x)) + \Delta^{n+1}\left(\alpha_{n+1}\binom{x}{n+1}\right)$$

or

$$\Delta^{n+1}(A_{n+1}(x)) = 0 + \alpha_{n+1} \implies \Delta^{n+1}(A_{n+1}(0)) = \alpha_{n+1},$$

from which, on account that $A_{n+1}(x)$ is a constant polynomial, $\Delta^{n+1}(a_0) = \alpha_{n+1}$ follows. Thus, $A_{n+1}(x) = A_n(x)+\Delta^{n+1}(a_0)\binom{x}{n+1}$ and, therefore, $A_{n+1}(n+1) = A_n(n+1)+\Delta^{n+1}(a_0)\binom{n+1}{n+1}$ or $a_{n+1} = b_n + \Delta^{n+1}(a_0)$, from which it follows that $b_n = a_{n+1} - \Delta^{n+1}(a_0)$.

Since for all $n \in \mathbb{N}$ we ahve

$$\Delta^{n+1}(a_0) = \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}a_{n+1-k},$$

then we have

$$\begin{aligned}
A_n(n+1) = b_n &= a_{n+1} - \Delta^{n+1}(a_0) \\
&= a_{n+1} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}a_{n+1-k} \\
&= \sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}a_{n+1-k}
\end{aligned}$$

and

$$A_n(x) = a_0 + \sum_{k=1}^{n} \Delta^k(a_0)\binom{x}{k} = \sum_{k=0}^{n} \Delta^k(a_0)\binom{x}{k}. \qquad \square$$

Remark 1. *As it is well-known, for any function $f(x)$ the difference operator Δ is defined by $\Delta f(x) = f(x+1) - f(x)$ and the k-times iterated difference operator Δ^k is defined recursively by $\Delta^0 f(x) = f(x)$ and $\Delta^k f(x) = \Delta(\Delta^{k-1} f(x))$ for $k \in \mathbb{N}$. Since $\Delta^0(c) = 0$, $\Delta(\binom{x}{1}) = \Delta(x) = 1$ and $\Delta(\binom{x}{n}) = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}$, then*

$$\Delta^k\left(\binom{x}{n}\right) = \Delta^k\left(\binom{x}{n-k}\right) = \begin{cases} 0 & \text{if } k > n, \\ 1 & \text{if } k = n. \end{cases}$$

A natural generalization of Problem 1 is the following:

Problem 2. *For a given sequence $a_0, a_1, \ldots, a_n, \ldots$, let $A_{m,n}(x)$, $n \geq 0$, be a polynomial of degree at most n such that $A_{m,n}(k) = a_{m+k}$ for $0 \leq k \leq n$. Find the value of $A_{m,n}(n+1)$.*

The answer is obvious and we have

$$\begin{aligned} A_{m,n}(n+1) &= a_{m+n+1} - \Delta^{n+1}(a_n) \\ &= a_{m+n+1} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} + a_{m+n+1-k} \\ &= \sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k} + a_{m+n+1-k} \end{aligned}$$

and

$$A_{m,n}(x) = a_m + \sum_{k=1}^{n} \Delta^k(a_m)\binom{x}{k} = \sum_{k=0}^{n} \Delta^k(a_m)\binom{x}{k}.$$

2 Applications

In what follows, some applications of the above results are given. We begin with the following.

Problem 3 (IMO Short List 1981). *Let $P(x)$ be a polynomial of degree n such that*

$$P(k) = 1 \Big/ \binom{n+1}{k}, \quad \text{for } 0 \le k \le n.$$

Find $P(n+1)$.

Solution. Using the correlation

$$A_n(n+1) = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} a_{n+1-k}$$

for $a_k = 1/\binom{n+1}{k} = 1/\binom{n+1}{n+1-k} = a_{n+1-k}$, we obtain

$$\begin{aligned} P(n+1) = A_n(n+1) &= \sum_{k=1}^{n+1} (-1)^{k-1} \left[\binom{n+1}{k} \Big/ \binom{n+1}{n+1-k} \right] \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} = \frac{(-1)^n + 1}{2}. \end{aligned}$$

□

The next application appeared in [1] and it is stated as follows:

Problem 4. *Let $A(x)$ be a polynomial with integer coefficients such that for $1 \le k \le n+1$ holds:*

$$A(k) = 5^k.$$

Find the value of $A(n+2)$.

Solution. We will solve a more general problem replacing 5 with any $a \neq 1$. Let $a_k = a^{k+1}$ $(0 \le k \le n)$, $A(x) = A_n(x)$ and $A(n+2) = A_n(n+1)$. Since

$$\begin{aligned} \Delta^k(a_0) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{k-i} = \sum_{i=0}^{k} \binom{k}{i} a^{k-i+1} \\ &= a \sum_{i=0}^{k} (-1)^i \binom{k}{i} a^{k-i} = a(a-1)^k, \end{aligned}$$

then

$$A_n(x) = a\sum_{k=0}^{n}(a-1)^k\binom{x}{k}$$

and

$$\begin{aligned} A(n+2) = A_n(n+1) &= a^{n+2} - \sum_{k=0}^{n+1}(-1)^{k-1}\binom{n+1}{k}a^{n+2-k} \\ &= a(a^{n+1}-(a-1)^{n+1}). \end{aligned}$$

Finally, setting $a = 5$ we get $A(n+2) = 5(5^{n+1} - 4^{n+1})$. $\square$

Remark 2. *Note that, if* $a_n = \alpha a^n + \beta b^n$, *then*

$$A_n(x) = \sum_{k=0}^{n}\big(\alpha a(a-1)^k + \beta b(b-1)^k\big)$$

and $A(n+1) = \alpha a(a^{n+1} - (a-1)^{n+1}) + \beta b(b^{n+1} - (b-1)^{n+1})$.

Finally, we close this paper by giving an application involving Fibonacci numbers.

Problem 5. *Let* $\{f_n\}_{n\geq 0}$ *be the Fibonacci sequence defined by* $f_0 = 0$, $f_1 = 1$, *and* $f_{n+1} = f_n + f_{n-1}$ *for all* $n \geq 1$. *Let* $F_{m,n}(x)$ *be a polynomial of degree at most* n *such that* $F_{m,n}(k) = f_{m+k}$ *for* $0 \leq k \leq n$. *Determine* $F_{m,n}(n+1)$.

Solution. First note that $\Delta(f_n) = f_{n+1} - f_n = f_{n-1}$. Then, $\Delta^k(f_n) = f_{n-k}$, $k \leq n$. But what happens if $k > n$? To get the answer to this question we need to extend the definition of Fibonacci sequence to negative values of n. We may define $f_{-n} = (-1)^{n+1}f_n$, as it is well-known. On account of the preceding we have

$$F_{m,n}(n+1) = f_{m+n+1} - \Delta^{n+1}(f_m) = f_{m+n+1} - f_{m-n-1}$$

and

$$F_{m,n}(x) = f_m + \sum_{k=1}^{n}\binom{x}{k}\Delta^k(f_m) = \sum_{k=0}^{n}\binom{x}{k}\Delta^k(f_m) = \sum_{k=0}^{n}\binom{x}{k}f_{m-k}.$$

In particular, if $m = 1$ we get

$$F_{1,n}(n+1) = f_{n+2} - f_{-n} = f_{n+2} - (-1)^{n+1}f_n = f_{n+2} + (-1)^n f_n$$

and

$$F_{1,n}(x) = \sum_{k=0}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} (-1)^k f_{k-1}.$$

□

References

[1] "Problem 64". *Mathproblems* 4.1 (2014), p. 244. ISSN: 2217-446X.

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